The Goldstone model and the Higgs mechanism

10 marca 2014 14:35

The Goldstone model

$$\mathcal{L}(x) = \partial^\mu \phi^* \partial_\mu \phi - \mu^2 |\phi|^2 - \lambda |\phi|^4 , \quad V(\phi) = \mu^2 |\phi|^2 + \lambda |\phi|^4 \quad \lambda > 0 \text{ (stability)}$$

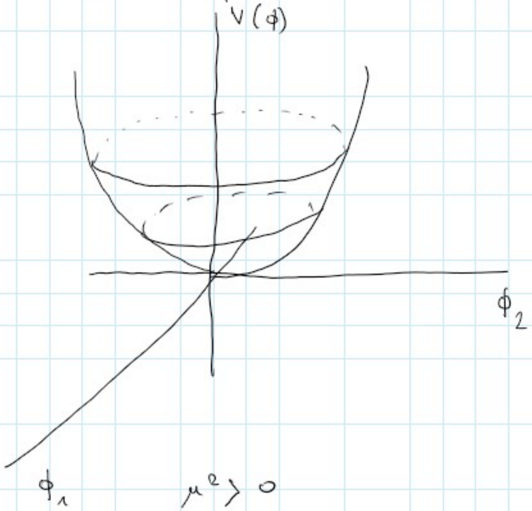
symmetry: $\phi \to e^{i\alpha} \phi$

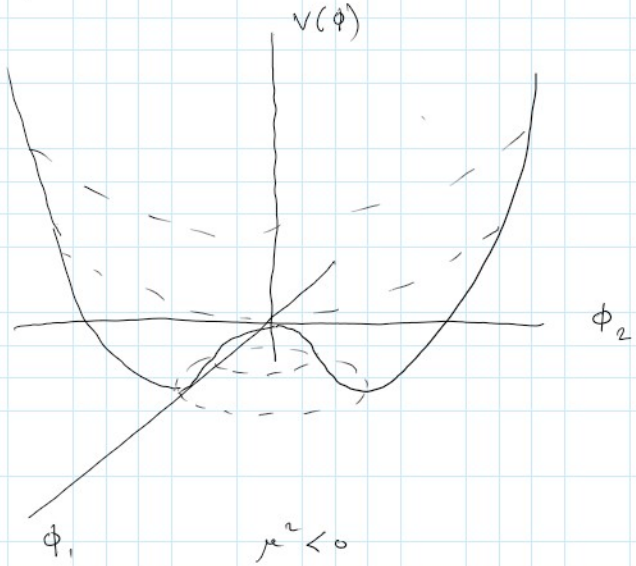
$$\mathcal{H}(x) = (\dot{\phi})^2 + (\bar{\nabla}\phi)^2 + V(\phi) \qquad E[\phi] = \int d^3x \, \mathcal{H}(x)$$

ground state $\Leftrightarrow$ minimum of $E[\phi]$ (keeping $\frac{\delta S}{\delta \phi} = 0$)

to minimize $E[\phi]$ we consider $\dot{\phi} = \nabla \phi = 0$, $\phi = \frac{1}{\sqrt{2}} (\phi_1 + i \phi_2)$

then $\frac{\partial V}{\partial \phi} = 0$





$\mu^2 < 0 \Rightarrow$ spontaneous symmetry breaking

$$\frac{\partial V}{\partial \phi} = \frac{\partial}{\partial \phi}\left(\mu^2 \phi^* \phi + \lambda \phi^{*2} \phi^2\right) = \phi^* \left(\mu^2 + 2\lambda \phi^* \phi\right) = \phi^* \left(\mu^2 + 2\lambda |\phi|^2\right) = 0$$

$$|\phi_{min}|^2 = \frac{-\mu^2}{2\lambda} \to \phi_{min} = \left(\frac{-\mu^2}{2\lambda}\right)^{1/2} e^{i\theta_{min}}$$

we can choose $\theta_{min} = 0$, so

$$\phi_{min} = \left(\frac{-\mu^2}{2\lambda}\right)^{1/2} \equiv \frac{v}{\sqrt{2}} \qquad v = \left(-\frac{\mu^2}{\lambda}\right)^{1/2} \to v^2\lambda = -\mu^2$$

$$\boxed{\mu^2 + \lambda v^2 = 0}$$

Let's expand around ϕ_{min}: $\phi = \phi_{min} + \frac{1}{\sqrt{2}}(\sigma(x) + i\eta(x))$

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\sigma\partial^\mu\sigma + \partial_\mu\eta\partial^\mu\eta\right) - \mu^2\frac{1}{2}\left[(v+\sigma)^2 + \eta^2\right] - \lambda\frac{1}{4}\left[(v+\sigma)^2 + \eta^2\right]^2 =$$

$$= \text{——} \;\text{"}\; \text{——} \;\text{"}\; \text{——} - \frac{\mu^2}{2}\left(v^2 + 2v\sigma + \sigma^2 + \eta^2\right) - \frac{\lambda}{4}\left(v^2 + 2v\sigma + \sigma^2 + \eta^2\right)^2 =$$

$$= \text{——} \;\text{"}\; \text{——} \;\text{"}\; \text{——} + \text{const} + \sigma\underbrace{\left(-v\mu^2 - \frac{\lambda}{\cancel{4}}\cancel{4}v^3\right)}_{=0} + \sigma^2\underbrace{\underbrace{\left(-\frac{\mu^2}{2} - \frac{\lambda}{\cancel{4}_2}\cancel{2}v^2 - \frac{\lambda}{\cancel{4}}\cancel{4}v^2\right)}_{\frac{\lambda}{\cancel{2}}\cancel{v^2} - \frac{\lambda}{\cancel{2}}\cancel{v^2} - \lambda v^2}}_{-\frac{1}{2}\underbrace{(2\lambda v^2)}_{m_\sigma^2}\sigma^2} + \eta^2\underbrace{\left(-\frac{\mu^2}{2} - \frac{\lambda}{\cancel{4}_2}\cancel{2}v^2\right)}_{=0} + \dots =$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{1}{2}m_\sigma^2\sigma^2 + \frac{1}{2}\partial_\mu\eta\partial^\mu\eta - \lambda v\sigma(\sigma^2 + \eta^2) - \frac{\lambda}{4}(\sigma^2 + \eta^2)^2$$

- η is massless!
- η corresponds to modes (fluctuations) that propagate along the minimum

The Higgs mechanism – the Abelian case

$$\mathcal{L} = (D_\mu\phi)^* D^\mu\phi - \mu^2|\phi|^2 - \lambda|\phi|^4 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$\phi(x) \xrightarrow{U(1)} \phi'(x) = e^{-iqf(x)}\phi(x) \qquad D_\mu\phi \equiv (\partial_\mu + iqA_\mu)\phi$$

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu f(x)$$

$$\mu^2 < 0 \Rightarrow \text{SSB} \qquad \phi(x) = \frac{1}{\sqrt{2}}(v + \sigma + i\eta)$$

$$\mu^2 < 0 \Rightarrow \text{SSB} \qquad \phi(x) = \frac{1}{\sqrt{2}}(v + \sigma + i\eta)$$

$$(D_\mu \phi)^* D^\mu \phi = \left(\partial_\mu - iqA_\mu\right)\frac{1}{\sqrt{2}}(v+\sigma-i\eta)\left(\partial^\mu + iqA^\mu\right)\frac{1}{\sqrt{2}}(v+\sigma+i\eta) = \dots - iqA_\mu \frac{v}{\sqrt{2}}\, iqA^\mu \frac{v}{\sqrt{2}} + \dots$$

$$= \dots + \frac{1}{2}(qv)^2 A_\mu A^\mu + \dots + \frac{1}{\sqrt{2}}\partial_\mu(\cancel{\sigma}-i\eta)\, iqA^\mu \frac{1}{\sqrt{2}} v - iqA_\mu \frac{v}{\sqrt{2}}\partial^\mu(\cancel{\sigma}+i\eta) + \dots =$$

$$= \frac{1}{2}\underbrace{(qv)^2}_{m_A^2} A_\mu A^\mu + (qv)\,\partial_\mu \eta A^\mu + \dots$$

$$\mathcal{L} = \frac{1}{2}\partial^\mu\sigma\partial_\mu\sigma - \frac{1}{2}m_\sigma^2\sigma^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m_A^2 A_\mu A^\mu + \partial_\mu\eta\partial^\mu\eta + m_A\partial_\mu\eta A^\mu + \dots$$

Reparameterization of $\phi(x)$: $\quad \phi(x) = \frac{1}{\sqrt{2}}(v+\sigma(x))\exp\left\{i\frac{\xi(x)}{v}\right\} = \frac{1}{\sqrt{2}}\left(v + \sigma(x) + i\xi(x) + \dots\right\}$

for $\sigma, \xi \ll v$ it is the same as in the Goldstone model

$$D_\mu\phi = (\partial_\mu + iqA_\mu)\frac{1}{\sqrt{2}}(v+\sigma)e^{i\xi/v} = \frac{1}{\sqrt{2}}\left(\partial_\mu\sigma e^{i\xi/v} + i\sigma\frac{\partial_\mu\xi}{v}e^{i\xi/v} + iqA_\mu(v+\sigma)e^{i\xi/v} + \cancel{v}i\frac{\partial_\mu\xi}{\cancel{v}}e^{i\xi/v}\right) =$$

$$= \frac{e^{i\xi/v}}{\sqrt{2}}\left[\partial_\mu\sigma + i\left(\frac{\sigma}{v}\partial_\mu\xi + q(v+\sigma)A_\mu + \partial_\mu\xi\right)\right] =$$

$$(D_\mu\phi)^* D^\mu\phi = \frac{1}{2}\left|\partial_\mu\sigma + i\left(\partial_\mu\xi + \frac{\sigma}{v}\partial_\mu\xi + qA_\mu(v+\sigma)\right|^2 =$$

$$= \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\left[\partial_\mu\xi\partial^\mu\xi + \left(\frac{\sigma}{v}\right)^2\partial_\mu\xi\partial^\mu\xi + q^2A_\mu A^\mu(v+\sigma)^2 + 2\frac{\sigma}{v}\partial_\mu\xi\partial^\mu\xi + 2q\frac{\sigma}{v}(q+\sigma)\partial_\mu\xi A^\mu + 2q\partial_\mu\xi A^\mu(v+\sigma)\right] =$$

$$\approx \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\xi\partial^\mu\xi + \frac{1}{2}(qv)^2A_\mu A^\mu + \text{interactions}$$

ξ – massless

– $\xi(x)$ could be eliminated from $\mathcal{L}$ by the following gauge transformation

– $\xi(x)$ could be eliminated from $\mathcal{L}$ by the following gauge transformation

$$\phi(x) \to \phi'(x) = e^{-i\frac{\xi(x)}{v}}\phi(x) = \frac{1}{\sqrt{2}}(v+\sigma(x))$$

$$\frac{\xi(x)}{v} = q f(x)$$

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu \underbrace{\xi(x)\frac{1}{qv}}_{f(x)}$$

$$\mathcal{L}(x) = \mathcal{L}[\phi(x), A_\mu(x)] = \mathcal{L}[\phi'(x), A'_\mu(x)] = \mathcal{L}\Big[\tfrac{1}{\sqrt{2}}(v+\sigma(x)), \underbrace{A'_\mu(x)}_{\downarrow\,(\text{notation change})\; A_\mu(x)}\Big]$$

$$\mathcal{L}(x) = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\underbrace{m_\sigma^2}_{(-2\mu^2)}\sigma^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\underbrace{m_A^2}_{(qv)^2}A_\mu A^\mu - \lambda v\sigma^3 - \frac{\lambda}{4}\sigma^4 + \frac{1}{2}q^2\sigma(2v+\sigma)A_\mu A^\mu$$

now: 1 real scalar — 1; 1 massive vector — 3; total 4

before: 2 real scalar — 2; 1 massless vector — 2; total 4

The non-Abelian case

complex doublet of SU(2): $\phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$

$$\mathcal{L} = (D_\mu\phi)^\dagger(D^\mu\phi) - V(\phi) - \frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu}$$

where

$$D_\mu\phi = (\partial_\mu - igT_a A^a_\mu)\phi \qquad T_a \equiv \frac{\sigma_a}{2}$$

where

$$D_\mu \phi = (\partial_\mu - i g T_a A^a_\mu)\,\phi \qquad T_a \equiv \frac{\sigma_a}{2}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\,\epsilon^{abc} A^b_\mu A^c_\nu$$

$$V(\phi) = +\mu^2(\phi^\dagger\phi) + \lambda(\phi^\dagger\phi)^2$$

For $\mu^2 < 0$ the minimum of $V(\phi)$ is at

$$\phi^\dagger_{min}\,\phi_{min} = \frac{v^2}{2} \qquad \text{with } v = \left(\frac{-\mu^2}{\lambda}\right)^{1/2}$$

we choose the vacuum in the form

$$\langle\phi\rangle \equiv \phi_{min} = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}$$

we define $\varphi = \phi - \langle\phi\rangle$, note that $\langle\varphi\rangle = 0$

$$(D_\mu\phi)^\dagger(D^\mu\phi) = \left[(\partial_\mu - i g T_a A^a_\mu)(\varphi + \langle\phi\rangle)\right]^\dagger\left[(\partial^\mu - i g T_b A^{b\mu})(\varphi + \langle\phi\rangle)\right]$$

Problem: Derive the mass-matrix for A^a_μ.

$$\Downarrow$$

$$\frac{1}{4} g^2 A^a_\mu A^{b\mu} \left(\sigma_a\langle\phi\rangle\right)^\dagger\left(\sigma_b\langle\phi\rangle\right) = \frac{1}{4} g^2 A^a_\mu A^{b\mu} \langle\phi\rangle^\dagger \underbrace{\sigma_a\sigma_b}_{\delta_{ab}\mathbb{1} + i\epsilon_{abc}\sigma_c}\langle\phi\rangle = \frac{1}{2}\left(\frac{gv}{2}\right)^2 A^a_\mu A^{a\mu}$$

$$\Downarrow$$

$$M_A = \frac{gv}{2}\mathbb{1}$$

In the scalar sector we have

$$\phi^\dagger\phi = (\varphi + \langle\phi\rangle)^\dagger(\varphi + \langle\phi\rangle) = \varphi^\dagger\varphi + \langle\phi\rangle^\dagger\varphi + \varphi^\dagger\langle\phi\rangle + \langle\phi\rangle^\dagger\langle\phi\rangle$$

$$(\phi^\dagger\phi)^2 = v^2\varphi^\dagger\varphi + \underbrace{(\langle\phi\rangle^\dagger\varphi + \varphi^\dagger\langle\phi\rangle)^2}_{\frac{v^2}{2}(\varphi_2 + \varphi_2^*)^2} + \cdots$$

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

$$\varphi = \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} \qquad \hookrightarrow \frac{v^2}{2}\left(\varphi_2 + \varphi_2^*\right)^2$$

$$\langle\phi\rangle^\dagger \varphi = \left(0, \frac{v}{\sqrt2}\right)\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = \frac{v}{\sqrt2}\varphi_2 \; ; \quad \varphi^\dagger\langle\phi\rangle = \left(\varphi_1^*, \varphi_2^*\right)\begin{pmatrix}0\\ \frac{v}{\sqrt2}\end{pmatrix} = \varphi_2^*\frac{v}{\sqrt2}$$

$$V(\phi) = \mu^2 \phi^\dagger\phi + \lambda\left(\phi^\dagger\phi\right)^2 = \underbrace{\left(\mu^2 + \lambda v^2\right)}_{\to 0}\left(|\varphi_1|^2 + |\varphi_2|^2\right) + \lambda\frac{v^2}{2}\left(\varphi_2 + \varphi_2^*\right)^2 + \cdots = \underbrace{-\frac{\mu^2}{2}\left(\varphi_2+\varphi_2^*\right)^2}_{} + \cdots$$

$$v^2 = -\frac{\mu^2}{\lambda}$$

$$-\frac{\mu^2}{2}4\left(\mathrm{Re}\,\varphi_2\right)^2 = \frac{1}{2}\underbrace{\left(-2\mu^2\right)}_{=m_h^2}\left(\sqrt2\,\mathrm{Re}\,\varphi_2\right)^2$$

$\sqrt2\,\mathrm{Re}\,\varphi_2$ - normalized physical Higgs field

$\mathrm{Im}\,\varphi_2$, $\mathrm{Re}\,\varphi_1$, $\mathrm{Im}\,\varphi_2$ - massless would-be-Goldstone bosons

A^1_μ, A^2_μ, A^3_μ - massive vector bosons

$m_h = \sqrt{2(-\mu^2)}$ - the Higgs boson mass

Question: what is the symmetry of the vacuum?

$\boxed{\text{The unitary gauge (non-Abelian case)}}$

Parameterization of the doublet: $\phi(x) = e^{i\sigma_a\frac{\xi^a(x)}{v}}\begin{pmatrix}0\\ \frac{v+\eta(x)}{\sqrt2}\end{pmatrix}$

We choose the following gauge transformation

$$\phi(x) \to \phi^u(x) = U(x)\,\phi(x) = \begin{pmatrix}0\\ \frac{v+\eta(x)}{\sqrt2}\end{pmatrix}$$

$$\frac{\sigma_a}{2}A^a_\mu \to \frac{\sigma_a}{2}B^a_\mu = U(x)\frac{\sigma_a}{2}A^a_\mu U^{-1}(x) - \frac{i}{g}\left[\partial_\mu U(x)\right]U^{-1}(x)$$

with

$$U(x) = e^{-i\frac{\sigma_a}{v}\cdot\xi_a(x)}$$

Then

$$D_\mu \phi \to U D_\mu \phi = \underbrace{U D_\mu U^{-1}}_{\partial_\mu - ig\frac{\sigma_a}{2}B^a_\mu} \phi^u = D^u_\mu \phi^u$$

$$D_\mu \phi = \left(\partial_\mu - ig\frac{\sigma_a}{2}A^a_\mu\right)\phi$$

$$U D_\mu U^{-1} = U\left(\partial_\mu - ig\frac{\sigma_a}{2}A^a_\mu\right)U^{-1} = \partial_\mu + U\partial_\mu U^{-1} - ig\, U\frac{\sigma_a}{2}A^a_\mu U^{-1} = \partial_\mu + U\partial_\mu U^{-1} - ig\left(\frac{\sigma_a}{2}B^a_\mu + \frac{i}{g}(\partial_\mu U)U^{-1}\right) =$$

$$= \partial_\mu + \underbrace{U\partial_\mu U^{-1} + (\partial_\mu U)U^{-1}}_{\partial_\mu(UU^{-1}) = 0} - ig\frac{\sigma_a}{2}B^a_\mu =$$

$$= \partial_\mu - ig\frac{\sigma_a}{2}B^a_\mu$$

$$\mathcal{L} = (D^a_\mu \phi^u)^+(D^{u\mu}\phi^u) - \underbrace{V(\phi^u)}_{} - \frac{1}{4}G^a_{\mu\nu}G^{a\,\mu\nu}$$

$$\phi^+\phi \to \phi^u U^{-1} U \phi^u = [\phi^u]^2$$

$$= \frac{\mu^2}{2}(v+\eta)^2 + \frac{\lambda}{4}(v+\eta)^4$$

$$G^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + g\epsilon^{abc}B^b_\mu B^c_\nu$$

$$(D^a_\mu \phi^u)^+(D^{u\mu}\phi^u) \longrightarrow \frac{g^2}{8}(0, v)\,\tau^a\tau^b\begin{pmatrix}0\\v\end{pmatrix}B^a_\mu B^{b\mu} = \frac{1}{2}\left(\frac{gv}{2}\right)^2 B^a_\mu B^{a\mu}$$

three massive gauge bosons